Heisenberg's Statistical Theory of Turbulence and the Equations of Motion for a Turbulent Flow *

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It is shown that consistent with Heisenberg's statistical theory of turbulence, equations of motion for a turbulent flow can be derived. These equations are linear integro-differential equations expressing the non-local interaction of eddies with different wave numbers on the basis of Heisenberg's statistical theory.

The nonlocal terms in these equations of motion for turbulent flow have to be determined from the energy spectrum of the turbulent motion, which is obtained from a nonlinear integro-differential equation. A general solution of the linear equations of motion is obtained by an arbitrary superposition of solutions. However, only those linear superpositions are permitted which are self-consistent with the energy spectrum of turbulent motion.

In contrast to the Navier-Stokes equations, the non-linearity occurs here only in the equation for the energy spectrum and not in the equations of motion itself. This fact greatly facilitates the integration of these equations.

Our analysis is extended to include turbulent convection. In the spirit of Heisenberg's hypothesis, equations of motion and energy are formulated which are consistent with the equations of the energy spectrum for free turbulent convection derived by Ledeoux, Schwarzschild and Spiegel. With this method, one may treat turbulent convection problems which arise in stellar and planetary atmospheres where the classical solution of laminar free convection cannot be applied.

1. Turbulent Fluid Motion and the Hypothesis of Isotropic Homogeneous Turbulence

We will assume that by some degree of approximation the motion of a turbulent fluid can be thought of as being a small scale turbulent velocity superimposed on the average fluid velocity. The "eddies" of the turbulent velocity field will interact and thus give rise to an additional frictional force which might be thought of as being caused by an eddy viscosity.

This picture is in someway analogous to the kinetic theory description of a gas or fluid. There also the fluid motion may be described by a small scale molecular motion superimposed onto an average fluid motion. The average velocity which defines a convenient frame of reference in which the fluid is at rest, is obtained by taking the velocity moment of the molecular velocity distribution function. The molecular velocity, superimposed on this average velocity, is then determined by the distribution function in this frame of reference.

As an example, one may consider the case of uniform fluid motion. If the fluid is in thermodynamic equilibrium, the distribution function in the frame at rest will be a displaced Maxwellian. Taking the velocity moment of this distribution function, one obtains the average fluid motion. A Galilei transformation to a system moving with this average velocity will then define the system at rest with the fluid. The distribution function in this new system is a homogeneous Maxwellian.

In fluid dynamics, the assumption of a homogeneous Maxwellian is only valid in very special cases, uniform motion being a trivial one. It will not be valid, in flow with velocity gradients, where momentum is exchanged by particle motion between different fluid layers. As a result the distribution function changes from a Maxwellian to some non-equilibrium distribution function. However, the relaxation time, being essentially the particle collision time, is generally so short that a deviation of the distribution function from a Maxwellian will be very small. Also, near a boundary the distribution function will be affected by collisons with the wall. But as long as the mean free path is small compared with the characteristic distance determined by the spatial separation of the boundaries, the deviation of the distribution function from a Maxwellian is always negligible. The assumption of a Maxwellian permits one to obtain simple expressions for the viscous friction force in the Navier-Stokes equation.

In the problem of turbulent motion, it is tempting to ask whether a similar approximation cannot be

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made there too, by looking at the analogy between colliding molecules in kinetic theory and interacting eddies in turbulent mixing. Because of the analogy between the two different processes in kinetic theory, the assumption of an isotropic Maxwellian velocity distribution can only correspond to a turbulent motion with an isotropic spectrum of turbulence. If we make this assumption the turbulent fluid motion may be described by an average velocity over which a small scale isotropic turbulent motion is superimposed.

The validity of an isotropic Maxwellian velocity distribution in kinetic theory resulted from a mean free path much smaller than the characteristic length of the fluid. A small mean free path implies many collisons by which the fluid relaxes rapidly into a Maxwellian distribution, and the assumption of a small mean free path is usually satisfied for most problems of interest.

From the analogy between colliding particles and interacting eddies, it follows that the assumption of a small mean free path in kinetic theory must correspond in turbulence to the assumption of a small "mean free path" for a turbulent eddy, small if compared with the dimensions of the system, in which the turbulent motion takes place. However, the assumption of a small "mean free path" for turbulent eddies cannot be satisfied as well as the corresponding assumption of a small mean free path in kinetic theory. The reason for this can be seen as follows: Turbulent motion can be thought of as consisting of eddies of different sizes described by a spectrum in wave number space. According to Heisenberg's theory the "mean free path" of an eddy with a wave number k is of the order $1/k$. If the characteristic dimension of the system is given by L, then the assumption of a small "mean free path" will break down for eddies with a size comparable to or larger than L, or for wave numbers smaller than $k = k_c = 1/L$, where wave number k_c is of the same order of magnitude as the cutoff wave number in Heisenberg's statistical theory of homogeneous isotropic turbulence.

2. Equations of Motions and Energy Spectrum of Turbulence

We are considering an incompressible fluid with the equation of motion

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\frac{1}{\varrho} \nabla p + \nu \nabla^2 \boldsymbol{v}, \quad (2.1)$$

and the equation of continuity

$$\operatorname{div} \boldsymbol{v} = 0 . \quad (2.2)$$

In Eq. (2.1) the term $(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v}$ represents a nonlinear interaction and it is this interaction which leads to turbulence.

In HEISENBERG's statistical theory [1], the behavior of isotropic homogeneous turbulence is described by an equation for the energy spectrum. Heisenberg's theory was generalized by CHANDRASEKHAR [2] to include time dependence. In this theory, the energy spectrum $F(k, t)$ of turbulent motion is given by the solution of an integro-differential equation:

$$-\frac{\partial}{\partial t} \int_{k_0}^{k} F(k', t)\, \mathrm{d}k'$$
$$= 2\left[\nu + \varkappa \int_{k}^{\infty} \sqrt{(F(k'', t)/k''^3)}\, \mathrm{d}k''\right] \int_{k_0}^{k} F(k', t)\, k'^2\, \mathrm{d}k',$$
$$k = (k_x^2 + k_y^2 + k_z^2)^{1/2} . \quad (2.3)$$

In Eq. (2.3) the second term in the square bracket is a function of the wave number k and may be considered as a turbulent or eddy viscosity, which is the fundamental assumption of Heisenberg's theory. The constant $\varkappa$ is a universal dimensionless constant of the order 1. Thus, one defines the eddy viscosity $\nu(k)$ in wave number space by

$$\nu(k) = \varkappa \int_{k}^{\infty} \sqrt{F(k', t)/k'^3}\, \mathrm{d}k' , \quad (2.4)$$

k_0 is the cutoff wave number which can be related to the size of the largest possible eddy consistent with the fluid boundaries.

To understand the physical meaning of Heisenberg's theory, we have to derive an equation for the energy spectrum from the equations of motion (2.1) assuming isotropic homogeneous turbulence. For this we expand the velocity and pressure fields in wave number space

$$\boldsymbol{v}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} \boldsymbol{v}(\boldsymbol{k}, t)\, e^{i \boldsymbol{k} \cdot \boldsymbol{r}}, \quad (2.5)$$

$$p(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} p(\boldsymbol{k}, t)\, e^{i \boldsymbol{k} \cdot \boldsymbol{r}}. \quad (2.6)$$

The assumption of isotropy implies that we can average the expressions for the energy spectrum over a spherical surface in $\boldsymbol{k}$-space. The energy

[1] W. HEISENBERG, Z. Phys. **124**, 628 [1948] and Proc. Roy. Soc. London A **195**, 402 [1948].
[2] S. CHANDRASEKHAR, Phys. Rev. **75**, 896 [1949].

spectrum $F(k, t)$ is then given by

$$F(k,t) = (V/8\pi^3)\, k^2 \int \langle \boldsymbol{v}(\boldsymbol{k},t)\, \boldsymbol{v}^*(\boldsymbol{k},t)\rangle \,\mathrm{d}\Omega\,. \quad (2.7)$$

In Eq. (2.7), V is a normalization volume and the brackets denote ensemble averages.

The equation for the energy spectrum is then obtained by taking the Fourier transform of Eq.(2.1) and multipling it by the complex conjugate transform of the velocity $\boldsymbol{v}^*(\boldsymbol{k}, t)$, and in accordance with (2.7) taking the average over a spherical surface in $\boldsymbol{k}$-space. We finally take into account the vanishing of pressure velocity correlations in the case of isotropic turbulence [3] and obtain

$$-\partial F/\partial t = 2\nu k^2 F - \int_{k_0}^{\infty} Q(k,k')\,\mathrm{d}k'\,. \quad (2.8)$$

In Eq. (2.8), Q results from the nonlinear interaction term and is trilinear in the Fourier transform of the velocity field. By comparing Eq. (2.3) with Eq. (2.8), it follows that Heisenberg's theory implies the ad hoc assumption that

$$\int_{k_0}^{k} \mathrm{d}k' \int_{k_0}^{\infty} Q(k',k'')\,\mathrm{d}k'' = - 2\varkappa \int_{k}^{\infty} \sqrt{F(k'')/k''^3}\,\mathrm{d}k'' \int_{k_0}^{k} F(k')\,k'^2\,\mathrm{d}k'\,. \quad (2.9)$$

Although it is not obvious how good this assumption really is, we can say that the overall agreement of Heisenberg's theory with measured energy spectra supports the hypothesis (2.9).

We are therefore inclined to ask the following question: What kind of equation must replace (2.1) in order that condition (2.9) is fulfilled *exactly ?* The answer to this question must lead to a set of equations of motion constistent with Heisenberg's statistical theory and which describe the motion of a turbulent flow.

To obtain the answer we proceed as follows:

I. We introduce an eddy viscosity in wave number space defined by $\nu_\varepsilon(k)$. This eddy viscosity is different from the eddy viscosity (2.4) but related to it by

$$\nu_\varepsilon(k) = \frac{1}{k^2 F(k)}\,\frac{\mathrm{d}}{\mathrm{d}k}\left[\nu(k)\int_{k_0}^{k} F(k')\,k'^2\,\mathrm{d}k'\right]. \quad (2.10)$$

II. With the eddy viscosity defined by Eq. (2.10) we construct the function

$$K(|\boldsymbol{r}|) = (2\pi)^{-3/2} \int \nu_\varepsilon(k)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{k}\,. \quad (2.11)$$

[3] T. O. Hinze, Turbulence, McGraw-Hill, New York 1959.

III. The equation of motion (2.1) is then replaced by the following equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\varrho}\nabla p + \nu \nabla^2 \boldsymbol{v} + (2\pi)^{-3/2}\int K(|\boldsymbol{r}-\boldsymbol{r}'|)\,\nabla^2 \boldsymbol{v}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'\,. \quad (2.12)$$

Equation (2.12) will lead exactly to Heisenberg's expression for the energy spectrum, which may easily be demonstrated. For the proof we take the Fourier transform of Eq. (2.12) and (2.2)

$$\frac{\partial \boldsymbol{v}(\boldsymbol{k})}{\partial t} = -i\,\frac{p(k)}{\varrho}\,\boldsymbol{k} - [\nu + \nu_\varepsilon(k)]\,k^2\,\boldsymbol{v}(\boldsymbol{k})\,, \quad (2.13)$$

$$\boldsymbol{k}\cdot\boldsymbol{v}(\boldsymbol{k}) = 0\,. \quad (2.14)$$

By taking the scalar product of (2.13) with $\boldsymbol{k}$, because of (2.14), results in

$$p(k) = 0\,, \quad (2.15)$$

hence

$$\frac{\partial \boldsymbol{v}(k)}{\partial t} = -[\nu + \nu_\varepsilon(k)]\,k^2\,\boldsymbol{v}(\boldsymbol{k})\,. \quad (2.16)$$

We multiply Eq. (2.13) with $\boldsymbol{v}^*(\boldsymbol{k})$, average according to (2.7) and have

$$\frac{\partial F}{\partial t} = 2[\nu + \nu_\varepsilon(k)]\,k^2 F\,. \quad (2.17)$$

Integrating Eqs. (2.17) from $k' = k_0$ to $k' = k$ we obtain Eq. (2.3). This completes the proof.

By comparing Eq. (2.12) with Eq. (2.1), one can see that the term $(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}$ has been replaced by a nonlocal term. The nonlocality is a result of the wave-number-dependent eddy viscosity. This behavior is not surprising because the eddies have a finite size of the order $1/k$ and therefore their interaction with other eddies must be nonlocal. Furthermore, in contrast to Eq. (2.1) Eq. (2.12) is linear. However, the nonlinearity appears here in the equation for the energy spectrum, the solution of which has to be known in order to construct the kernel function $K(|\boldsymbol{r}|)$ according to Eq. (2.11). Since the equation for the energy spectrum can be only solved after the flow problem is determined, one has to solve the system of Eqs. (2.12), (2.14) and (2.7) self-consistently. The removal of the nonlinearity from the equations of motion is a great advantage over the Navier Stokes equations. The linearity in the equations of motions simplifies greatly the solution of the set of equations describing the motion of turbulent flow. This will be demonstrated for the important problem of free turbulent convection treated below.

3. Turbulent Convection

The theory outlined in the preceding paragraph can be easily generalized to include turbulent convection. This may be of significance for the treatment of convection problems in stellar and planetary atmospheres.

Heisenberg's theory is valid only for fluid motions of small Mach numbers. Therefore, in applying Heisenberg's theory to convection problems we are restricted to the Boussinesq approximation [4] with the following equations of motion

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = - \frac{1}{\varrho} \nabla p' + \nu \nabla^2 \boldsymbol{v} + g\, \alpha\, T' \boldsymbol{e}_z\,, \tag{3.1}$$

and the energy equation

$$\frac{\partial T'}{\partial t} + \boldsymbol{v} \cdot \nabla T' = \chi \nabla^2 T' + \beta\, \boldsymbol{v} \cdot \boldsymbol{e}_z\,, \tag{3.2}$$

to be supplemented by

$$\operatorname{div} \boldsymbol{v} = 0\,. \tag{3.3}$$

In the Eqs. (3.1)–(3.2), T' and p' are the perturbations of the temperature and pressure fields. α is the thermal expansion coefficient, $\beta = \nabla \triangle T$ is the excess of the temperature gradient over the adiabatic temperature gradient, and $\boldsymbol{e}_z$ is a unit vector in the vertical direction. χ is the heat conduction coefficient.

To find the implications of Heisenberg's concept on the set of Eqs. (3.1)–(3.3), we again have to derive the equations for the turbulent energy spectrum. There we need in addition to $F(k, t)$ defined by (2.7), the following spectral functions

$$G(k,t) = (V/8\pi^3)\, k^2 \int \langle T'(\boldsymbol{k},t)\, T'^*(\boldsymbol{k},t) \rangle \, \mathrm{d}\Omega\,, \tag{3.4}$$

$$H(k,t) = \tfrac{1}{2} (V/8\pi^3)\, k^2\, \boldsymbol{e}_z \cdot \int \{\langle \boldsymbol{v}(\boldsymbol{k},t)\, T^{*\prime}(\boldsymbol{k},t)\rangle + \langle \boldsymbol{v}^*(\boldsymbol{k},t)\, T'(\boldsymbol{k},t)\rangle\} \, \mathrm{d}\Omega\,. \tag{3.5}$$

By the same procedure as before, we obtain from (3.1) and (3.2) the following equations for the spectral functions

$$- \frac{\partial F}{\partial t} = 2\, \nu\, k^2 F - 2\, g\, \alpha\, H - \int_{k_0}^{\infty} Q(k, k')\, \mathrm{d}\, k'\,, \tag{3.6}$$

$$- \frac{\partial G}{\partial t} = 2\, \chi\, k^2 G - 2\, \beta\, H - \int_{k_0}^{\infty} U(k, k')\, \mathrm{d}k'\,. \tag{3.7}$$

In Eq. (3.7), $U(k, k')$ is mixed trilinear in temperature and velocity. This set of equations, in order to have a solution, must be supplemented by an additional relation between the functions F, G and H. In the theory of LEDOUX, SCHWARZSCHILD and SPIEGEL [5] it is assumed that the velocity and temperature fluctuations are in phase which is expressed by

$$H = \sqrt{\tfrac{1}{2} F G}\,. \tag{3.8}$$

To formulate the equations for the spectral functions in the spirit of Heisenberg's eddy viscosity hypothesis it is rather obvious that we have to make the same approximation for $Q(k,k')$ as in (2.9). For the third term on the r.h.s. of Eq. (3.7), however, we have to make an additional hypothesis not contained in Heisenberg's theory. By comparing Eq. (3.6) and (3.7) one may connect the third term on the r.h.s of (3.7) with the heat transported by eddies. It is thus tempting to put in analogy to Eq. (2.9).

$$\int_{k_0}^{k} \mathrm{d}k' \int_{k_0}^{\infty} U(k', k'')\, \mathrm{d}k'' = - 2\, \chi(k) \int_{k_0}^{\infty} G(k')\, k'^2\, \mathrm{d}k'\,, \tag{3.9}$$

where $\chi(k)$ is a wave-number-dependent eddy heat conduction coefficient.

The heat conduction coefficient for an ideal gas is related to the kinematic viscosity ν by

$$\chi = \nu/\gamma\,, \tag{3.10}$$

where γ is the specific heat ratio.

Depending on the number of degrees of freedom of the gas molecules, which are between 3 and 6, we have

$$4/3 < \gamma < 5/3\,. \tag{3.11}$$

If the temperature and velocity fluctuations are in phase as assumed by Ledoux, Schwarzschild and Spiegel, then a proportionality similar to (3.10) should also hold for the eddies describing the turbulent velocity field. This implies

$$\chi(k) = a\, \nu(k)\,, \quad 3/5 < a < 3/4\,. \tag{3.12}$$

We thus have

$$\int_{k_0}^{k} \mathrm{d}k' \int_{k_0}^{\infty} U(k', k'')\, \mathrm{d}k'' = - 2\, a\, \nu(k) \int_{k_0}^{k} G(k')\, k'^2\, \mathrm{d}k'\,. \tag{3.13}$$

The r.h.s. of Eq. (3.13) has the same property as U, that is, it is bilinear in the temperature and linear in the velocity fluctuation.

[4] E. A. SPIEGEL and G. VERONIS, Astrophys. J. **131**, 442 [1960].

[5] P. LEDOUX, M. SCHWARZSCHILD and E. A. SPIEGEL, Astrophys. J. **133**, 184 [1961].

In analogy to (2.10) we introduce a second turbulent heat conduction coefficient $\chi(k)$ defined

$$\chi_\varepsilon(k) = \frac{1}{k^2\,G(k)}\,\frac{\mathrm{d}}{\mathrm{d}k}\,[\chi(k)\int_{k_0}^{k} G(k')\,k'^2\,\mathrm{d}k'] = \frac{a}{k^2\,G(k)}\,\frac{\mathrm{d}}{\mathrm{d}k}\,[\nu(k)\int_{k_0}^{k} G(k')\,k'^2\,\mathrm{d}k'] \tag{3.14}$$

hence, we obtain for the spectral functions F, G and H the following set of equations

$$-\frac{\partial F}{\partial t} = 2\{\nu + \nu_\varepsilon(k)\}\,k^2 F - 2\,g\,\alpha\,H\,,$$
$$-\frac{\partial G}{\partial t} = 2\{\chi + \chi_\varepsilon(k)\}\,k^2 G - 2\,\beta\,H\,, \tag{3.15}$$
$$H = \sqrt{\tfrac{1}{2}\,F\,G}\,.$$

In stellar atmospheres the heat transport by radiation is much larger than the heat transport by turbulence. For this reason, the heat transport by convection can be neglected as was done in the theory of Ledoux, Schwarzschild and Spiegel. In their treatment it was furthermore assumed that $\partial/\partial t = 0$, which means restriction to steady state solutions. The second Eq. (3.15) is then approximated by

$$\chi\,k^2 G = \beta\,H\,, \tag{3.16}$$

or by eliminating G from (3.16) with the third Eq. (3.15), we have

$$H = (\beta/2\,\chi\,k^2)\,F\,. \tag{3.17}$$

Finally inserting expression (3.17) into the first equation of (3.15) one obtains the equation by Ledoux, Schwarzschild and Spiegel for the energy spectrum of free turbulent convection.

As in the preceding chapter it is now easy to construct equations of motions for turbulent convection which are consistent with the equations for the spectral functions (3.15). In analogy to (2.11), we introduce the function

$$K^*(|\boldsymbol{r}|) = (2\,\pi)^{-3/2}\int \chi_\varepsilon(k)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{k}\,, \tag{3.18}$$

and replace (3.1) and (3.2) by the following new set of equations

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\varrho}\,\nabla\,p' + \nu\,\nabla^2\,\boldsymbol{v} + (2\,\pi)^{-3/2}\int K(|\boldsymbol{r}-\boldsymbol{r}'|)\,\nabla^2\,\boldsymbol{v}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' + g\,\alpha\,T'\,\boldsymbol{e}_z\,, \tag{3.19}$$

$$\frac{\partial T'}{\partial t} = \chi\,\nabla^2\,T' + (2\,\pi)^{-3/2}\int K^*(|\boldsymbol{r}-\boldsymbol{r}'|)\,\nabla^2\,T'(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' + \beta\,\boldsymbol{v}\cdot\boldsymbol{e}_z\,. \tag{3.20}$$

The first two equations of (3.15) for the spectral functions F, G and H follow from (3.19) and (3.20). The proof is straightforward and similar to the derivation of Eq. (2.3) from (2.12).

4. Selfconsistent Solution for Steady State Free Turbulent Convection

In the classical treatment of thermal convection, the Eqs. (3.1) and (3.2) are linearized by omitting the nonlinear terms $(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}$ and $\boldsymbol{v}\cdot\nabla T'$. The resulting set of equations is then Fourier-analyzed in space and time into the following principal modes of convection ($\boldsymbol{v} = \{u, v, w\}$, V_0 velocity amplitude)

$$u = +\,V_0\,\frac{k_z\,k_x}{k^2}\,\cos(k_x\,x)\,\sin(k_y\,y)\,\cos(k_z\,z)\,e^{nt}\,,$$
$$v = +\,V_0\,\frac{k_z\,k_y}{k^2}\,\sin(k_x\,x)\,\cos(k_y\,y)\,\cos(k_z\,z)\,e^{nt}\,, \tag{4.1}$$
$$w = +\,V_0\,\frac{k_x^2+k_y^2}{k^2}\,\sin(k_x\,x)\,\sin(k_y\,y)\,\sin(k_z\,z)\,e^{nt}\,,$$
$$T' = +\,V_0\,\frac{n+\nu k^2}{g\alpha}\,\sin(k_x\,x)\,\sin(k_y\,y)\,\sin(k_z\,z)\,e^{nt}\,,$$
$$p' = -\,V_0\,\varrho\,\frac{n+\nu k^2}{k}\,\frac{k_z}{k}\,\sin(k_x x)\sin(k_y y)\cos(k_z z)\,e^{nt}\,.$$

The growth rate n for these principal modes is determined by the following characteristic equation or dispersion relation:

$$n = -\,\frac{\nu+\chi}{2}\,k^2\left[1 \pm \left(1 - \mu + \mu\,\frac{g\,\alpha\,\beta}{\nu\,\chi}\,\frac{k_x^2+k_y^2}{k^6}\right)^{1/2}\right], \tag{4.2}$$

where $$\mu = \frac{4\,\nu\,\chi}{(\nu+\chi)^2} = \frac{4\,Pr}{(1+Pr)^2}\,. \tag{4.3}$$

$Pr = \nu/\chi$ is the so-called Prandtl number.

The dispersion Eq. (4.2) has two solutions corresponding to two possible signs. This means Eq. (4.1) describes two modes, stable and unstable. The stable modes are interpreted physically by falling currents, and the unstable modes by rising currents.

In addition to the two modes given by Eq. (4.1), the equations of motions permit one more mode in which only horizontal velocities occur. This mode is represented by

$$u = +\,V_0\,\frac{k_y}{k}\,\cos(k_x\,x)\,\sin(k_y\,y)\,\cos(k_z\,z)\,e^{nt}\,,$$
$$v = -\,V_0\,\frac{k_x}{k}\,\sin(k_x\,x)\,\cos(k_y\,y)\,\cos(k_z\,z)\,e^{nt}\,,$$
$$w = T' = p' = 0\,, \tag{4.4}$$

and the dispersion relation determining the growth rate is

$$n = -\nu k^2 . \tag{4.5}$$

Of special interest are the steady state solutions for which $n = 0$. It is obvious that the condition $n = 0$ can be fulfilled only in Eq. (4.2) for the minus sign and by putting

$$\nu \chi k^6 = g \alpha \beta (k_x^2 + k_y^2) ,$$

resp.

$$\nu \chi (k_x^2 + k_y^2 + k_z^2)^3 = g \alpha \beta (k_x^2 + k_y^2) . \tag{4.6}$$

Eq. (4.6) represents an algebraic surface of the sixth order in $\boldsymbol{k}$-space. The steady state solutions obtained from Eq. (4.1) by putting $n = 0$ are always restricted to this surface.

The solution given by the classical theory is only applicable to laminar convection problems. However, since most convection phenomena in atmospheres are turbulent, the classical theory breaks down. The turbulent equations of motion which have been derived in the preceding chapter can be used to calculate turbulent convection problems in an approximation consistent with Heisenberg's theory.

In the turbulent problem we will restrict ourselves for the sake of simplicity to steady state solutions for which we always have $\partial/\partial t = 0$. This was also done in the special case considered by Ledoux, Schwarzschild and Spiegel. The restriction to time independent solutions results in a considerable mathematical simplification of the posed problem. The reason is that in the time dependent case the eddy transport coefficients ν_ε and χ_ε are also time dependent, since the spectral functions F, G and H from which they are computed are time dependent. As a result, the Eqs. (3.19) and (3.20) in regard to their time behavior are here not differential equations with constant coefficients. The time behavior of the solutions is therefore no longer simply given by an exponential dependence as in the case of laminar convection.

In treating the steady state turbulent convection problem, we put everywhere $\partial/\partial t = 0$ and perform a Fourier transform of Eq. (3.19), (3.20) together with the supplementary Eq. (3.3) into $\boldsymbol{k}$-space. The solution of the turbulent problem is then obtained as follows: We introduce a total viscosity $\nu^*(k)$ in $\boldsymbol{k}$-space which is defined by

$$\nu^*(k) = \nu + \nu_\varepsilon(k) , \tag{4.7}$$

and similarly a total heat conduction coefficient

$$\chi^*(k) = \chi + \chi_\varepsilon(k) . \tag{4.8}$$

With these wave number dependent coefficients, we obtain the following principal modes of steady state turbulent convection

$$\begin{aligned}
u &= + V_0(k) \frac{k_z k_x}{k^2} \cos(k_x x) \sin(k_y y) \cos(k_z z) , \\
v &= + V_0(k) \frac{k_z k_y}{k^2} \sin(k_x x) \cos(k_y y) \cos(k_z z) , \\
w &= + V_0(k) \frac{k_x^2 + k_y^2}{k^2} \sin(k_x x) \sin(k_y y) \sin(k_z z), \qquad (4.9) \\
T' &= + V_0(k) \frac{\nu^* k^2}{g \alpha} \sin(k_x x) \sin(k_y y) \sin(k_z z) , \\
p' &= - V_0(k) \varrho \nu^* k_z \sin(k_x x) \sin(k_y y) \cos(k_z z) .
\end{aligned}$$

We will call the set of Eq. (4.9), the basic set of solutions for free turbulent convection. Because of the linearity of the equations of motion and energy, any linear superposition of solutions from the basic set (4.9) form a possible solution but only those linear combinations of solutions which also satisfy Eq. (2.7) and (3.4) will give a selfconsistent solution. In agreement with the assumption of isotropic turbulence the linear superposition of the basic solutions (4.9) shall only depend upon the total wave number k, which can be achieved by making the amplitude factor in Eq. (4.9) a function of k, $V_0 = V_0(k)$, which has to be determined by inserting (4.9) into (2.7) and (3.4) and solving for $V_0(k)$. This however, can be done only after the equations for the spectral functions F, G and H have been solved.

The solutions (4.9) have furthermore to obey the following compatibility condition

$$\nu^*(k) \chi^*(k) k^6 = g \alpha \beta (k_x^2 + k_y^2) , \tag{4.10}$$

which is the turbulent analogue to Eq. (4.6), which was the condition for laminar steady state convection. As in the case of Eq. (4.6), Eq. (4.10) represents a surface in $\boldsymbol{k}$-space to which steady state solutions are restricted. The character of this surface is known if $\nu^*(k)$ and $\chi^*(k)$ have been determined from the equations for the spectral functions F, G and H.

We would like to remark that the solutions (4.9) for $n = 0$ after replacing ν and χ by ν^* and χ^*, have the same form as the corresponding laminar solutions (4.1).

The equations for the spectral functions F, G and H are simplified in the time independent case.

After introducing the definitions (4.7) and (4.8), they are reduced to the following set of equations

$$\nu^* k^2 F = g\alpha H, \quad \chi^* k^2 G = \beta H, \quad H = \sqrt{\tfrac{1}{2} F G}. \quad (4.11)$$

Multiplying both sides of the first and second Eq. (4.11) and eliminating H with the help of the third equation, results in

$$2\nu^* \chi^* k^4 = g\alpha\beta. \quad (4.12)$$

The form of the surface, Eq. (4.10) is now easily determined by eliminating the product $\nu^* \chi^*$ with the help of Eq. (4.12). The result is given by

$$k_x^2 + k_y^2 = \tfrac{1}{2} k^2, \quad \text{resp.} \quad k_x^2 + k_y^2 = k_z^2, \quad (4.13)$$

The solutions for steady state free turbulent convection are hence restricted to a cone, the equation of which is given by (4.13). We would like to remark that the same condition, Eq. (4.13), was used in the theory of Ledoux, Schwarzschild and Spiegel and was derived in a more intuitive way by assuming that the steady state turbulent convection is primarily caused by the laminar modes of convection with the largest growth rates.

Because of Eq. (4.13), we obtain for the basic set of solutions for free turbulent convection, Eq. (4.9), the following expressions

$$\begin{aligned}
u &= + V_0(k) \frac{k_x}{2\sqrt{k_x^2 + k_y^2}} \cos(k_x x) \sin(k_y y) \cos\left(\sqrt{k_x^2 + k_y^2}\, z\right), \\
v &= + V_0(k) \frac{k_y}{2\sqrt{k_x^2 + k_y^2}} \sin(k_x x) \cos(k_y y) \cos\left(\sqrt{k_x^2 + k_y^2}\, z\right), \\
w &= + \tfrac{1}{2} V_0(k) \sin(k_x x) \sin(k_y y) \sin\left(\sqrt{k_x^2 + k_y^2}\, z\right), \\
T' &= + V_0(k) \frac{2\nu^*(k_x^2 + k_y^2)}{g\alpha} \sin(k_x x) \sin(k_y y) \sin\left(\sqrt{k_x^2 + k_y^2}\, z\right), \\
p' &= - V_0(k)\, \varrho\, \nu^* \sqrt{k_x^2 + k_y^2} \sin(k_x x) \sin(k_y y) \cos\left(\sqrt{k_x^2 + k_y^2}\, z\right).
\end{aligned} \quad (4.14)$$

The value of ν^* which enters into (4.14) has to be determined from the equations for the spectral functions given by (4.11). In the theory of Ledoux, Schwarzschild and Spiegel, it was assumed that $\chi^* \cong \chi$. Hence, by inserting $\chi^* = \chi$ into Eq. (4.12) leads to exactly the same equation for $F(k)$ which was derived by these authors.

If $\chi^* \neq \chi$ we have to use a second equation obtained from (4.11), in order to eliminate $G(k)$ from (4.12) which enters into it through $\chi^*(k)$. From the first and the third Eq. (4.11), we obtain by eliminating H

$$G = (4\nu^{*2} k^4 / g^2 \alpha^2) F, \quad (4.15)$$

and therefore because of Eqs. (3.14) and (4.8), we find

$$\chi^* = \chi + \frac{a}{\nu^{*2} k^6 F} \frac{\mathrm{d}}{\mathrm{d}k} \left[\nu(k) \int_{k_0}^{k} F(k') \nu^{*2}(k') k'^6 \mathrm{d}k'\right]. \quad (4.16)$$

The integro-differential equation for $F(k)$ is then obtained by inserting Eq. (4.16) into (4.12):

$$2\nu^* \left\{\chi + \frac{a}{\nu^{*2} k^6 F} \frac{\mathrm{d}}{\mathrm{d}k}\left[\nu(k) \int_{k_0}^{k} F \nu^{*2} k'^6 \mathrm{d}k'\right]\right\} k^4 = g\alpha\beta. \quad (4.17)$$

5. Conclusion

Equations of motion for turbulent flow have been derived which are consistent with Heisenberg's statistical theory of turbulence in the sense that Heisenberg's equation for the energy spectrum is an exact consequence of them. In contrast to other phenomenological theories which make some kind of an *a priori* assumption concerning eddy transport coefficients, the theory presented in this paper is nonlocal, and as a consequence of the nonlocality the equations are integro-differential equations. The theory is extended to include turbulent convection.

We have also derived the general integro differential equation for the energy spectrum of free turbulent convection. The theory may be used to determine the character of complicated convection problems in stellar or planetary atmospheres which otherwise can be treated by numerical analysis only under immense computational efforts.

Finally it should be remarked that the demonstrated nonlocal character of the eddy viscosity suggests to describe turbulent flow problems by simply adding to the Navier-Stokes equations nonlocal terms with kernel functions which have to be determined experimentally. Such an approach has some similarity to the mixing length theories by Prandtl but may be better suited to approximate phenomena resulting from turbulent friction.